On the Self-Induction of Electric Currents in a Thin Anchor-Ring.

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In their useful compendium of "Formulæ and Tables for the Calculation of Mutual and Self-Inductance,"* Rosa and Cohen remark upon a small discrepancy in the formulæ given by myself† and by M. Wien‡ for the self-induction of a coil of circular cross-section over which the current is *uniformly distributed.* With omission of n, representative of the number of windings, my formula was

$$L = 4\pi a\left[\log\frac{8a}{\rho}-\frac{7}{4}+\frac{\rho^2}{8a^2}\left(\log\frac{8a}{\rho}+\frac{1}{3}\right)\right], \tag{1}$$

where ρ is the radius of the section and a that of the circular axis. The first two terms were given long before by Kirchhoff.§ In place of the fourth term within the bracket, viz., $+\frac{1}{24}\rho^2/a^2$, Wien found $-\cdot 0083\rho^2/a^2$. In either case a correction would be necessary in practice to take account of the space occupied by the insulation. Without, so far as I see, giving a reason, Rosa and Cohen express a preference for Wien's number. The difference is of no great importance, but I have thought it worth while to repeat the calculation and I obtain the same result as in 1881. A confirmation after 30 years, and without reference to notes, is perhaps almost as good as if it were independent. I propose to exhibit the main steps of the calculation and to make extension to some related problems.

The starting point is the expression given by Maxwell‖ for the mutual induction M between two neighbouring co-axial circuits. For the present purpose this requires transformation, so as to express the inductance in terms of the situation of the elementary circuits relatively to the circular axis. In the figure, O is the centre of the circular axis, A the centre of a section B through the axis of symmetry, and the position of any point P of the section is given by polar co-ordinates relatively to A, viz., by PA (ρ)

* 'Bulletin of the Bureau of Standards, Washington,' 1908, vol. 3, No. 1.

† 'Roy. Soc. Proc.,' 1881, vol. 32, p. 104; 'Scientific Papers,' vol. 2, p. 15.

‡ 'Ann. d. Physik,' 1894, vol. 53, p. 934; it would appear that Wien did not know of my earlier calculation.

§ 'Pogg. Ann.,' 1864, vol. 121, p. 551.

‖ 'Electricity and Magnetism,' § 705.

and by the angle PAC (ϕ). If ρ_1, ϕ_1; ρ_2, ϕ_2, be the co-ordinates of two points of the section P_1, P_2, the mutual induction between the two circular circuits represented by P_1, P_2 is approximately

$$\frac{M_{12}}{4\pi a} = \left\{1+\frac{\rho_1\cos\phi_1+\rho_2\cos\phi_2}{2a}+\frac{\rho_1^2+\rho_2^2+2\rho_1^2\sin^2\phi_1+2\rho_2^2\sin^2\phi_2}{16a^2}\right.$$

$$\left.-\frac{2\rho_1\rho_2\cos(\phi_1-\phi_2)+4\rho_1\rho_2\sin\phi_1\sin\phi_2}{16a^2}\right\}\log\frac{8a}{r}$$

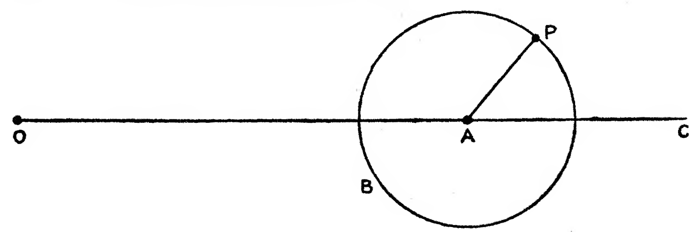
$$-2-\frac{\rho_1\cos\phi_1+\rho_2\cos\phi_2}{2a}$$

$$+\frac{3(\rho_1^2+\rho_2^2)-4(\rho_1^2\sin^2\phi_1+\rho_2^2\sin^2\phi_2)+2\rho_1\rho_2\cos(\phi_1-\phi_2)}{16a^2}, \quad (2)$$

in which r, the distance between P_1 and P_2, is given by

$$r^2 = \rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\phi_1-\phi_2). \quad (3)$$

Further details will be found in Wien's memoir; I do not repeat them because I am in complete agreement so far.



For the problem of a current uniformly distributed we are to integrate (2) twice over the area of the section. Taking first the integrations with respect to ϕ_1, ϕ_2, let us express

$$\frac{1}{4\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi}\frac{M_{12}}{4\pi a}\,d\phi_1 d\phi_2, \quad (4)$$

of which we can also make another application. The integration of the terms which do not involve $\log r$ is elementary. For those which do involve $\log r$ we may conveniently replace ϕ_2 by $\phi_1+\phi$, where $\phi = \phi_2-\phi_1$, and take first the integration with respect to ϕ, ϕ_1 being constant. Subsequently we integrate with respect to ϕ_1.

It is evident that the terms in (2) which involve the first power of ρ vanish in the integration. For a change of ϕ_1, ϕ_2, into $\pi-\phi_1, \pi-\phi_2$, respectively reverses $\cos\phi_1$ and $\cos\phi_2$, while it leaves r unaltered. The definite integrals required for the other terms are*

* Todhunter's 'Int. Calc.,' §§ 287, 289.

$$\int_{-\pi}^{+\pi} \log(\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\phi)\,d\phi = \text{greater of } 4\pi\log\rho_2 \text{ and } 4\pi\log\rho_1, \tag{5}$$

$$\int_{-\pi}^{+\pi} \cos m\phi \log(\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\phi)\,d\phi$$
$$= -\frac{2\pi}{m} \times \text{smaller of } \left(\frac{\rho_2}{\rho_1}\right)^m \text{ and } \left(\frac{\rho_1}{\rho_2}\right)^m, \tag{6}$$

m being an integer. Thus

$$\frac{1}{4\pi^2}\iint \log r\,d\phi_1 d\phi_2 = \frac{1}{4\pi^2}\int_{-\pi}^{+\pi} d\phi_1 \int_{-\pi}^{+\pi} d\phi \log r = \text{greater of } \log\rho_2 \text{ and } \log\rho_1. \tag{7}$$

So far as the more important terms in (4)—those which do not involve ρ as a factor—we have at once

$$\log(8a) - 2 - \text{greater of } \log\rho_2 \text{ and } \log\rho_1. \tag{8}$$

If ρ_2 and ρ_1 are equal, this becomes

$$\log(8a/\rho) - 2. \tag{9}$$

We have now to consider the terms of the second order in (2). The contribution which these make to (4) may be divided into two parts. The first, not arising from the terms in $\log r$, is easily found to be

$$\frac{\rho_1^2+\rho_2^2}{8a^2}[\log(8a)+\tfrac{1}{2}]. \tag{10}$$

The difference between Wien's number and mine arises from the integration of the terms in $\log r$, so that it is advisable to set out these somewhat in detail. Taking the terms in order, we have as in (7)

$$\frac{1}{4\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} \log r\,d\phi_1 d\phi_2 = \text{greater of } \log\rho_2 \text{ and } \log\rho_1. \tag{11}$$

In like manner

$$\frac{1}{4\pi^2}\iint \sin^2\phi_1 \log r\,d\phi_1 d\phi_2 = \tfrac{1}{2}[\text{greater of } \log\rho_2 \text{ and } \log\rho_1], \tag{12}$$

and $\frac{1}{4\pi^2}\iint \sin^2\phi_2 \log r\,d\phi_1 d\phi_2$ has the same value. Also by (6), with $m = 1$,

$$\frac{1}{4\pi^2}\iint \cos(\phi_2-\phi_1)\log r\,d\phi_1 d\phi_2 = -\tfrac{1}{2}[\text{smaller of } \rho_2/\rho_1 \text{ and } \rho_1/\rho_2]. \tag{13}$$

Finally $\quad \frac{1}{4\pi^2}\iint \sin\phi_1 \sin\phi_2 \log r\,d\phi_1 d\phi_2$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{+\pi} d\phi_1 \sin\phi_1 \int_{-\pi}^{+\pi} (\sin\phi_1\cos\phi + \cos\phi_1\sin\phi)\log r\,d\phi$$
$$= -\tfrac{1}{4}[\text{smaller of } \rho_2/\rho_1 \text{ and } \rho_1/\rho_2]. \tag{14}$$

Thus altogether the terms in (2) of the second order involving $\log r$ yield in (4)

$$-\frac{\rho_1^2+\rho_2^2}{8a^2}[\text{greater of } \log\rho_2 \text{ and } \log\rho_1]-\frac{\rho_1\rho_2}{8a^2}\left[\text{smaller of } \frac{\rho_2}{\rho_1} \text{ and } \frac{\rho_2}{\rho_1}\right]. \quad (15)$$

The complete value of (4) to this order of approximation is found by addition of (8), (10), and (15).

By making ρ_2 and ρ_1 equal we obtain at once for the self-induction of a current limited to the circumference of an anchor-ring, and uniformly distributed over that circumference,

$$\mathrm{L} = 4\pi a\left[\left(1+\frac{\rho^2}{4a^2}\right)\log\frac{8a}{\rho}-2\right], \quad (16)$$

ρ being the radius of the circular section. The value of L for this case, when ρ^2 is neglected, was virtually given by Maxwell.*

When the current is uniformly distributed over the area of the section we have to integrate again with respect to ρ_1 and ρ_2 between the limits 0 and ρ in each case. For the more important terms we have from (8)

$$\frac{1}{\rho^4}\iint d\rho_1^2 d\rho_2^2\,[\log 8a-2-\text{greater of } \log\rho_2 \text{ and } \log\rho_1]$$

$$= \log 8a-2-\frac{1}{2\rho^4}\int d\rho_1^2\,[\log\rho_1^2\,.\,\rho_1^2+\rho^2(\log\rho^2-1)-\rho_1^2(\log\rho_1^2-1)]$$

$$= \log 8a-2-\log\rho+\frac{1}{4} = \log\frac{8a}{\rho}-\frac{7}{4}. \quad (17)$$

A similar operation performed upon (10) gives

$$\frac{\log(8a)+\frac{1}{2}}{8a^2\rho^4}\iint(\rho_1^2+\rho_2^2)\,d\rho_1^2\,d\rho_2^2 = \frac{\log(8a)+\frac{1}{2}}{8}\frac{\rho^2}{a^2}. \quad (18)$$

In like manner, the first part of (15) yields

$$-\frac{\rho^2}{16a^2}(\log\rho^2-\tfrac{1}{3}).$$

For the second part we have

$$-\frac{1}{8a^2\rho^4}\iint d\rho_1^2 d\rho_2^2\,[\text{smaller of } \rho_2^2, \rho_1^2] = -\frac{\rho^2}{24a^2};$$

thus altogether from (15)

$$-\frac{\rho^2}{8a^2}(\log\rho+\tfrac{1}{6}). \quad (19)$$

The terms of the second order are accordingly, by addition of (18) and (19),

$$\frac{\rho^2}{8a^2}\left(\log\frac{8a}{\rho}+\tfrac{1}{3}\right). \quad (20)$$

* 'Electricity and Magnetism,' §§ 692, 706.

To this are to be added the leading terms (17); whence, introducing $4\pi a$, we get finally the expression for L already stated in (1).

It must be clearly understood that the above result, and the corresponding one for a *hollow* anchor-ring, depend upon the assumption of a uniform distribution of current, such as is approximated to when the coil consists of a great number of windings of wire insulated from one another. If the conductor be solid and the currents due to induction, the distribution will, in general, not be uniform. Under this head Wien considers the case where the currents are due to the variation of a homogeneous magnetic field, parallel to the axis of symmetry, and where the distribution of currents is governed by *resistance*, as will happen in practice when the variations are slow enough. In an elementary circuit the electromotive force varies as the square of the radius and the resistance as the first power. Assuming as before that the whole current is unity, we have merely to introduce into (4) the factors

$$\frac{(a+\rho_1\cos\phi_1)(a+\rho_2\cos\phi_2)}{a^2}, \tag{21}$$

M_{12} retaining the value given in (2).

The leading term in (21) is unity, and this, when carried into (14), will reproduce the former result. The term of the first order in ρ in (21) is $(\rho_1\cos\phi_1+\rho_2\cos\phi_2)/a$, and this must be combined with the terms of order ρ^0 and ρ^1 in (2). The former, however, contributes nothing to the integral. The latter yield in (4)

$$\frac{\rho_1^2+\rho_2^2}{4a^2}\{\log 8a-1-\text{greater of }\log\rho_1\text{ and }\log\rho_2\}+\frac{\text{smaller of }\rho_1^2\text{ and }\rho_2^2}{4a^2}. \tag{22}$$

The term of the second order in (21), viz., $\rho_1\rho_2/a^2 \,.\, \cos\phi_1\cos\phi_2$, needs to be combined only with the leading term in (2). It yields in (4)

$$\frac{\text{smaller of }\rho_1^2\text{ and }\rho_2^2}{4a^2}. \tag{23}$$

If ρ_1 and ρ_2 are equal (ρ), the additional terms expressed by (22), (23), become

$$\frac{\rho^2}{2a^2}\log\frac{8a}{\rho}. \tag{24}$$

If (24), multiplied by $4\pi a$, be added to (16), we shall obtain the self-induction for a shell (of uniform infinitesimal thickness) in the form of an anchor-ring, the currents being excited in the manner supposed. The result is

$$L = 4\pi a\left\{\left(1+\frac{3\rho^2}{4a^2}\right)\log\frac{8a}{\rho}-2\right\}. \tag{25}$$

We now proceed to consider the solid ring. By (22), (23) the terms, additional to those previously obtained on the supposition that the current was uniformly distributed, are

$$\frac{1}{\rho^4}\iint d\rho_1{}^2 d\rho_2{}^2\left[\frac{\text{smaller of } \rho_1{}^2 \text{ and } \rho_2{}^2}{2a^2}\right.$$
$$\left.+\frac{\rho_1{}^2+\rho_2{}^2}{4a^2}\{\log 8a-1-\text{greater of } \log\rho_1 \text{ and } \log\rho_2\}\right]. \quad (26)$$

The first part of this is $\rho^2/6a^2$, and the second is $\frac{\rho^2}{4a^2}\{\log 8a-1-\log\rho+\frac{1}{6}\}$. The additional terms are accordingly

$$\frac{\rho^2}{4a^2}\left\{\log\frac{8a}{\rho}-\frac{1}{6}\right\}. \quad (27)$$

These multiplied by $4\pi a$ are to be added to (1). We thus obtain

$$\mathrm{L} = 4\pi a\left[\log\frac{8a}{\rho}-\frac{7}{4}+\frac{3\rho^2}{8a^2}\log\frac{8a}{\rho}\right] \quad (28)$$

for the self-induction of the solid ring when currents are slowly generated in it by uniform magnetic forces parallel to the axis of symmetry. In Wien's result for this case there appears an additional term within the bracket equal to $-0{\cdot}092\ \rho^2/a^2$.

A more interesting problem is that which arises when the alternations in the magnetic field are rapid instead of slow. Ultimately the distribution of current becomes independent of *resistance*, and is determined by induction alone. A leading feature is that the currents are *superficial*, although the ring itself may be solid. They remain, of course, symmetrical with respect to the straight axis, and to the plane which contains the circular axis.

The magnetic field may be supposed to be due to a current x_1 in a circuit at a distance, and the whole energy of the field may be represented by

$$\mathrm{T} = \tfrac{1}{2}\mathrm{M}_{11}x_1{}^2+\tfrac{1}{2}\mathrm{M}_{22}x_2{}^2+\tfrac{1}{2}\mathrm{M}_{33}x_3{}^2+\ldots+\mathrm{M}_{12}x_1x_2+\mathrm{M}_{13}x_1x_3+\ldots+\mathrm{M}_{23}x_2x_3$$
$$+\ldots, \quad (29)$$

x_2, x_3, etc., being currents in other circuits where no independent electromotive force acts. If x_1 be regarded as given, the corresponding values of x_2, x_3, ..., are to be found by making T a minimum. Thus

$$\left.\begin{aligned}\mathrm{M}_{12}x_1+\mathrm{M}_{22}x_2+\mathrm{M}_{32}x_3+\ldots &= 0,\\ \mathrm{M}_{13}x_1+\mathrm{M}_{23}x_2+\mathrm{M}_{33}x_3+\ldots &= 0,\end{aligned}\right\} \quad (30)$$

and so on, are the equations by which x_2, etc., are to be found in terms of x_1. What we require is the corresponding value of T′, formed from T by omission of the terms containing x_1.

The method here sketched is general. It is not necessary that x_2, etc., be currents in particular circuits. They may be regarded as generalised co-ordinates, or rather velocities, by which the kinetic energy of the system is defined.

For the present application we suppose that the distribution of current round the circumference of the section is represented by

$$\{\alpha_0+\alpha_1\cos\phi_1+\alpha_2\cos 2\phi_1+\ldots\}\frac{d\phi_1}{2\pi}, \tag{31}$$

so that the total current is α_0. The doubled energy, so far as it depends upon the interaction of the ring currents, is

$$\frac{1}{4\pi^2}\iint(\alpha_0+\alpha_1\cos\phi_1+\alpha_2\cos 2\phi_1+\ldots)(\alpha_0+\alpha_1\cos\phi_2+\ldots)\,\mathrm{M}_{12}d\phi_1 d\phi_2, \tag{32}$$

where M_{12} has the value given in (2), simplified by making ρ_1 and ρ_2 both equal to ρ. To this has to be added the double energy arising from the interaction of the ring currents with the primary current. For each element of the ring currents (31) we have to introduce a factor proportional to the area of the circuit, viz., $\pi(a+\rho\cos\phi_1)^2$. This part of the double energy may thus be taken to be

$$\mathrm{H}\int d\phi_1(a+\rho\cos\phi_1)^2(\alpha_0+\alpha_1\cos\phi_1+\alpha_2\cos 2\phi_1+\ldots),$$

that is

$$2\pi\mathrm{H}\,\{(a^2+\tfrac{1}{2}\rho^2)\,\alpha_0+a\rho\alpha_1+\tfrac{1}{4}\rho^2\alpha_2\}, \tag{33}$$

α_3, etc., not appearing. The sum of (33) and (32) is to be made a minimum by variation of the α's.

We have now to evaluate (32). The coefficient of α_0^2 is the quantity already expressed in (16). For the other terms it is not necessary to go further than the first power of ρ in (2). We get

$$4\pi a\left[\alpha_0^2\left\{\log\frac{8a}{\rho}\left(1+\frac{\rho^2}{4a^2}\right)-2\right\}+\tfrac{1}{4}(\alpha_1^2+\tfrac{1}{2}\alpha_2^2+\tfrac{1}{3}\alpha_3^2+\ldots)\right.$$
$$\left.+\frac{\rho}{a}\left\{\frac{\alpha_0\alpha_1}{2}\left(\log\frac{8a}{\rho}-1\right)+\frac{\alpha_1}{8}(2\alpha_0+\alpha_2)+\frac{\alpha_2}{2\,.\,8}(\alpha_1+\alpha_3)+\frac{\alpha_3}{3\,.\,8}(\alpha_2+\alpha_4)+\ldots\right\}\right], \tag{34}$$

Differentiating the sum of (33), (34), with respect to α_0, α_1, etc., in turn, we find

$$\mathrm{H}\,(a^2+\tfrac{1}{2}\rho^2)+4a\alpha_0\left\{\log\frac{8a}{\rho}\left(1+\frac{\rho^2}{4a^2}\right)-2\right\}+\rho\alpha_1\left(\log\frac{8a}{\rho}-\frac{1}{2}\right)=0, \tag{35}$$

$$\mathrm{H}\rho+\alpha_1+\frac{\rho}{a}\left\{\alpha_0\left(\log\frac{8a}{\rho}-\frac{1}{2}\right)+\frac{3\alpha_2}{8}\right\}=0, \tag{36}$$

$$\mathrm{H}\rho^2+2a\alpha_2+\rho\left(\frac{3\alpha_1}{2}+\frac{5\alpha_3}{6}\right)=0. \tag{37}$$

The leading term is, of course, α_0. Relatively to this, α_1 is of order ρ, α_2 of order ρ^2, and so on. Accordingly, α_2, α_3, etc., may be omitted entirely from (34), which is only expected to be accurate up to ρ^2 inclusive. Also, in α_1 only the leading term need be retained.

The ratio of α_1 to α_0 is to be found by elimination of H between (35), (36). We get

$$\frac{\alpha_1}{\alpha_0} = \frac{\rho}{a}\left\{3\log\frac{8a}{\rho} - \frac{15}{2}\right\}. \tag{38}$$

Substituting this in (34), we find as the coefficient of self-induction

$$\mathrm{L} = 4\pi a\left[\log\frac{8a}{\rho}\left(1+\frac{\rho^2}{4a^2}\right) - 2 + \frac{\rho^2}{4a^2}\left(3\log\frac{8a}{\rho} - \frac{15}{2}\right)\left(5\log\frac{8a}{\rho} - \frac{17}{2}\right)\right]. \tag{39}$$

The approximate value of α_0 in terms of H is

$$\alpha_0 = -\frac{\mathrm{H}a}{4\left(\log\frac{8a}{\rho} - 2\right)}. \tag{40}$$

A closer approximation can be found by elimination of α_1 between (35), (36).

In (39) the currents are supposed to be induced by the variation (in time) of an unlimited uniform magnetic field. A problem, simpler from the theoretical point of view, arises if we suppose the uniform field to be limited to a cylindrical space co-axial with the ring, and of diameter less than the smallest diameter of the ring $(2a-2\rho)$. Such a field may be supposed to be due to a cylindrical current sheet, the length of the cylinder being infinite. The ring currents to be investigated are those arising from the instantaneous abolition of the current sheet and its conductor.

If πb^2 be the area of the cylinder, (33) is replaced simply by

$$\mathrm{H}\int d\phi_1\, b^2(\alpha_0 + \alpha_1\cos\phi_1 + \ldots) = 2\pi \mathrm{H} b^2 \alpha_0. \tag{41}$$

The expression (34) remains unaltered and the equations replacing (35), (36), are thus

$$\mathrm{H}b^2 + 4a\alpha_0\left\{\log\frac{8a}{\rho}\left(1+\frac{\rho^2}{4a^2}\right) - 2\right\} + \rho\alpha_1\left(\log\frac{8a}{\rho} - \frac{1}{2}\right) = 0. \tag{42}$$

$$\alpha_1 + \frac{\rho}{a}\left(\log\frac{8a}{\rho} - \frac{1}{2}\right)\alpha_0 = 0. \tag{43}$$

The introduction of (43) into (34) gives for the coefficient of self-induction in this case—

$$\mathrm{L} = 4\pi a\left[\log\frac{8a}{\rho}\left(1+\frac{\rho^2}{4a^2}\right) - 2 - \frac{\rho^2}{4a^2}\left(\log\frac{8a}{\rho} - \frac{1}{2}\right)^2\right]. \tag{44}$$

It will be observed that the sign of α_1/α_0 is different in (38) and (43).

The peculiarity of the problem last considered is that the primary current occasions no magnetic force at the surface of the ring. The consequences were set out 40 years ago by Maxwell in a passage* whose significance was very slowly appreciated. "In the case of a current sheet of no resistance, the surface integral of magnetic induction remains constant at every point of the current sheet.

"If, therefore, by the motion of magnets or variations of currents in the neighbourhood, the magnetic field is in any way altered, electric currents will be set up in the current sheet, such that their magnetic effect, combined with that of the magnets or currents in the field, will maintain the normal component of magnetic induction at every point of the sheet unchanged. If at first there is no magnetic action, and no currents in the sheet, then the normal component of magnetic induction will always be zero at every point of the sheet.

"The sheet may therefore be regarded as impervious to magnetic induction, and the lines of magnetic induction will be deflected by the sheet exactly in the same way as the lines of flow of an electric current in an infinite and uniform conducting mass would be deflected by the introduction of a sheet of the same form made of a substance of infinite resistance.

"If the sheet forms a closed or an infinite surface, no magnetic actions which may take place on one side of the sheet will produce any magnetic effect on the other side."

All that Maxwell says of a current sheet is, of course, applicable to the surface of a perfectly conducting solid, such as our anchor-ring may be supposed to be. The currents left in the ring after the abolition of the primary current must be such that the magnetic force due to them is *wholly tangential* to the surface of the ring. Under this condition $\int_{-\pi}^{+\pi} \mathrm{M}_{12}\, d\phi_2$ must be independent of ϕ_1, and we might have investigated the problem upon this basis.

In Maxwell's notation α, β, γ, denote the components of magnetic force, and the whole energy of the field T is given by

$$\mathrm{T} = \frac{1}{8\pi}\iiint(\alpha^2+\beta^2+\gamma^2)\, dx\,dy\,dz = \tfrac{1}{2}\mathrm{L}\alpha_0^2. \tag{45}$$

Moreover α_0, the total current, multiplied by 4π is equal to the "circulation" of magnetic force round the ring. In this form our result admits of immediate application to the hydrodynamical problem of the circulation of incom-

* 'Electricity and Magnetism,' §§ 654, 655. Compare my "Acoustical Observations," 'Phil. Mag.,' 1882, vol. 13, p. 340 ; 'Scientific Papers,' vol. 2, p. 99.

pressible frictionless fluid round a solid having the form of the ring; for the components of velocity u, v, w, are subject to precisely the same conditions as are α, β, γ. If the density be unity, the kinetic energy T of the motion has the expression

$$T = \frac{L}{8\pi} \times (\text{circulation})^2, \tag{46}$$

L having the value given in (44).

P.S., March 4.—Sir W. D. Niven, who in 1881 verified some other results for self-induction—those numbered (11), (12) in the paper referred to—has been good enough to confirm the formulæ (1), (28) of the present communication, in which I differ from M. Wien.

The Diffusion and Mobility of Ions in a Magnetic Field.

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1. When the motion of ions in a gas takes place in a magnetic field the rates of diffusion and the velocities due to an electric force may be determined by methods similar to those given in a previous paper.*

The effect of the magnetic field may be determined by considering the motion of each ion between collisions with molecules. The magnetic force causes the ions to be deflected in their free paths, and when no electric force is acting the paths are spirals, the axes being along the direction of the magnetic force. If H be the intensity of the magnetic field, e the charge, and m the mass of an ion, then the radius r of the spiral is mv/He, v being the velocity in the direction perpendicular to H. The distance that the ion travels in the interval between two collisions in a direction normal to the magnetic force is a chord of the circle of radius r. The average lengths of these chords may be reduced to any fraction of the projection of the mean free path in the direction of the magnetic force, so that the rate of diffusion of ions in the directions perpendicular to the magnetic force is less than the rate of diffusion in the direction of the force.

In the previous paper it has been shown that the coefficient of diffusion K is a measure of the rate of increase of the mean square R^2 of the distance of any distribution from any point, dR^2/dt being equal to 6K. Since

* 'Roy. Soc. Proc.,' February, 1912, vol. 86.